MAKING MATHEMATICS VITAL

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Algebra revisited*

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Poor concepts of the symbols used in algebra contribute to students' difficulties. Some concerns include the understanding of the addition sign, the equals sign and the variety of meanings for the pronumeral x. Following a discussion of student understanding of algebraic concepts, some activities are suggested which foster discussion around some of the 'big ideas' of algebra and have the potential to make the concepts of algebra explicit.

Introduction

The mention of the word algebra often brings a negative reaction from the listener. Many adults comment that mathematics was 'okay' until they started algebra. It then became hard and sometimes they add that they subsequently failed mathematics. I have heard teachers comment that when the word algebra was mentioned it was like a chilly wind blew through the classroom. The perception seems to be that algebra is difficult.

Why is it that algebra causes so many difficulties for children learning mathematics? Many children seem to 'hit a brick wall' in their mathematics learning early in Years 7 and 8 and this is usually attributed to algebra. Recently there has been a lot of international attention on early algebra being introduced in the first few years of school. Changes in curriculum in many places have included algebraic development right from the start of schooling and this should make a difference but these changes will take some years to filter through and affect students at Years 7 and 8.

Understanding of the operation symbols

So what are some of the causes of these difficulties with learning algebra? One of the difficulties is that although one aspect of algebra is *generalised arithmetic*, the signs and symbols in algebra are not exactly the same as they are understood by many students in arithmetic. For example, when some students see 5 + 7 they immediately recognise the + as a sign to combine the two numbers and give the response 12. Once the number is seen as 12 the original components such as the + sign are no longer visible and a single number, 12, replaces the expression 5 + 7. In algebra however, $a + 7$ is different: the plus sign does not mean 'combine the two parts to make a single number in the same way' as

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it did for the arithmetic expression (Chalouh & Herscovics, 1988). The expression $a + 7$ can be considered as a single object made by combining the two components a and 7 but these components maintain their identity within the object.

Many children try a variety of ways to combine the separate components. Most teachers of Years 7 and 8 have seen expressions such as $4x + 3$ simplified by students to $7x$. Some students will have learned the procedure for simplifying expressions and can use it to simplify quite complex expressions but then add an extra step to write the final expression as a single term, thus eliminating the plus sign. Students who leave the expression as $4x + 3$ without trying to combine the parts are said to have 'acceptance of lack of closure' (Collis, 1975). This understanding is a critical part of algebraic development.

Understanding of '='

Another aspect of differences between symbol use in arithmetic and algebra is the equals (=) sign. Freudenthal (1983) claimed four different categories of meaning for the equals sign:

- the result of sum;
- quantitative sameness;
- a statement that something is true for all values of the variable (identity); and
- a statement that assigns a value to a new variable.

A full understanding of the equals sign as it is used in algebra requires all of these meanings. However, for many students = is the sign that indicates the need to do something — an operator sign — or to move to the next step; or even as an indicator of where to write the answer — a syntactical indicator (Carpenter & Levi, 2000) — so they will record incorrectly using equals signs. I saw this demonstrated in a classroom recently where students were solving a problem. The question concerned how many legs there were with two lions, four cubs and four storks. After the sharing time at the end of the lesson, the display shown below was on the blackboard.

$2 \times 4 = 8 + 4 \times 4$
$=8 + 16 = 24 + 4 \times 2$
$= 24 + 8 = 32$

This misuse of the equals sign during the solution to a problem is also common among secondary school students who use = as a sign to do the next step in solving an equation; for example, $\cos A = 0.5 = 60°$. Other students will use the equals sign at the start of a row, so it becomes the sign for the next line of a solution even if the task is solving an equation as in the example below.

$3x - 4 = 2x + 5$
$= 3x - 4 - 2x = 5$
$x - 4 = 5$
$= x = 9$

The meanings of these symbol components of arithmetic and algebra need to be made explicit for the students. The new curriculum in Queensland addresses this by including the algebraic structure as part of early understanding of mathematics and recognising that this structure underlies both arithmetic and algebra. In other places also, the primary school curriculum has recognised the need for improved understanding of the equals sign. However, for Year 7 and 8, students who have not had these experiences and who are still operating with the idea that the equals sign is an indication of where to write the answer, there needs to be discussion about these issues so that the reasons for using the symbols in a particular way are based on understanding rather than 'because the teacher tells me I have to set it out that way.'

Understanding of the pronumeral

Yet another group of errors arise because of lack of explicit explanation for the different uses of the pronumeral. Here is a list of equations where pronumerals can be considered to have different meanings.

1. $x + 2 = 5$
2. $3x + 4 = 15$
3. $x(x - 2) - 15 = 0$
4. $a(x + b) = ax + ab$
5. $2x + 3x = 5x$
6. $y = 2x - 4$
7. $4x + 3y = 12$
8. $y = mx + c$
9. $A = l \times w$

Sometimes the x represents a number which is known, and sometimes it represents an unknown number. Sometimes it represents one number and sometimes many. On some occasions the pronumeral is a variable, and on others, a constant. A more complete list of possible meanings for the 'x' is given here.

- a specific known number
- a specific unknown number
- more than one specific number
- any number
- (any object)
- a variable which may be dependent or independent
- a constant
- a quantity that can be measured
- a quantity that can be calculated

In the first of the equations above, most students look at it and know immediately that $x = 3$. In this situation x is not an unknown. The equation is transparent. Many students thus find it difficult to understand why the textbooks use complicated algorithms to 'solve' such equations. The methods of solution given make much more sense when applied to the second of the equations as nearly all students would need a formal method of solution. The third equation not only requires a method of solution but yields more than one value for the pronumeral.

The distributive law, which is the fourth equation, is an identity which is always true for all possible values of the pronumerals involved and relates to Freundenthal's third category of meaning for the equals sign. In the fifth equation, the x is not restricted to pronumerals or algebraic objects made up of pronumerals, but indeed could be any object. This is the root of what has become to be known as 'fruit salad algebra,' based on using the letter to represent an object often starting with that letter so $3a + 4a = 7a$ is read with the a being 'apples' rather than the desired understanding at this level of a representing a number.

The understanding of x which relates to functions and relations is as a variable, and is represented in equations six to eight. It also relates to Freudenthal's fourth category for the understanding of the equals sign. As an independent variable, x does not just represent any number but rather all numbers in the possible domain. For students there is a difference between an expression written in the assignation form such as equation 6 and the linear relation represented in equation 7. Equation 8 also raises the idea of constants and variables. I remember being puzzled over this distinction for years as a student in high school and at university.

In the final equation, the l and the w represent the length and width of a rectangle and as such in the student's eyes are know quantities because they are easily measured. The A is different because it is not measured directly but is rather calculated. This difference in understanding explains why students who otherwise can solve an equation like $3x = 21$, have difficulty finding the length when the area is 21 cm^2 and the width is 3 cm (Usiskin, 1988).

Students often adopt one meaning for the pronumerals and do not attend to others. A classic situation arose when a teacher was returning a test to a Year 8 student. The student complained that he had been unfairly treated as the teacher had marked the question wrong when it was correct. The teacher looked at the linear equation, for which the student had the answer 39 and explained to the student that 14 was the correct value as it made the equation correct. The student responded in frustration, 'But all last year you told us that x could be any number and so what is wrong with 39?' The symbol x has many different meanings which are rarely if ever made explicit and this can contribute to students' misunderstandings. These multiple meanings need to become part of the classroom conversation.

One key aspect of algebra is its use in generalisation. The student above has over generalised the meaning of the x but on other occasions we want students to generalise. Algebra has often been described as generalised arithmetic, and part of it is the abstraction from specifics in arithmetic to general underlying structures.

Difficulties in this abstraction process often occur because students may focus on inappropriate generalisations and interpretations, as well as obstructions caused by semantics and alternative approaches to semantics deduced from the 'concrete' situation. For example, given a simple one- or two-step linear equation in early algebra, students will often solve it by a guess and check method in spite of the teacher presenting a different approach. This is reinforced by success in the problems in the Year 7/8 textbooks, and becomes entrenched but does not lead to further understanding and allow transfer to more difficult situations. Similarly in arithmetic, the equality symbol is often seen as a signal to perform operations, but this is a limited conception and causes an obstacle in algebra. Left to their own devices without direction, students are unlikely to develop the semantics of algebra as we know them because the types of experiences they have are limited and often lead to alternative representations which do not then relate to other situations. Another example of students developing entrenched but non-productive understanding is with students using a backtracking method. They might record it happily as $5x + 3 = 8 = 5 = 1$ and all students involved at the time understand what this means, but it is a misuse or different meaning of the symbols and will limit future development. This means clear guidance is needed to assist the students to construct knowledge and use mathematical language and sign systems that are compatible with the language and sign systems of others.

Backtracking causes a further obstacle. Students are often shown how they can solve fairly complex equations with one occurrence of the variable on the left hand side of an equation and a single number on the right hand side. They practise this skill and become adept at using it. This often leads to a strong reluctance to relinquish it when in the following years they meet equations for which backtracking cannot be used, and thus handicaps their further development in algebra.

Algebra sense

Students need to develop a sense of algebra. What do I mean by algebra sense? Algebra sense is an understanding of the objects of algebra and the different representations as well as the ability to sense the form of the result of a particular process (Horne & Maurer, 1998). It is the ability to visualise the nature and form of the solution and to move readily between the representations or mathematical sign systems rather than the ability to work with the objects to produce the required solutions, although of course producing solutions is also necessary in developing algebra. In many ways, algebra sense corresponds to number sense, though algebraic experiences are not as much a part of the students' world as numerical experiences.

A critical part of developing algebra sense is encouraging discussion where the use of language and student explanation can assist them in their developing understanding. The few activities below are designed to allow all students to participate in developing mental algebraic skills and more particularly to make sense of algebra. The key part of these activities is the ensuing discussion in which the issues can be made explicit and the big ideas of algebra be raised. Part of the focus is on some of these key principles of algebra. For example, the first activity focusses on the approaches students use in solving equations. The idea is to enable the students to share their ideas about how to solve equations. The ensuing discussion should also raise the issue of when different methods are useful and efficient and the difference between 'arithmetic' linear equations which can be solved using backtracking methods and 'algebraic' equations which have more than one occurrence of the x. Filloy and Sutherland (1996) call this separation between what they see as arithmetic and algebraic, the didactic cut.

Activity 1

Which of these equations

A. are easy to solve in your head?

B. could you solve in your head but it requires extra thinking?

C. would you prefer to use a pen and paper to solve?

1. $2x + 5 = 9$
2. $3x - 4 = x + 2$
3. $4x + 3 = 12$
4. $5 = 2x + 1$
5. $3x - 8 = 5x + 2$
6. $6x - 5 = 3x + 2$
7. $3(x - 4) = x + 2$
8. $2(x + 5) = 9$
9. $5x - 2 = 9$
10. $(11x + 5)/3 - 4 + 2 \times 3 = 11$

Another question to ask then is what different methods could be used to solve these equations and which is the most efficient method for each question? We know many students use guess and test even though teachers have often tried to insist on the students setting out their solutions to equations by using a balance method. For this activity the key focus for the students could be which methods are most suitable for which equations. Instead of the question above about doing the problems in their heads, the question might be:

For which of these equations could you use
A. guess and test;
B. backtracking;
C. the balance method; or
D. other?

The activity can also involve group discussion before the whole class has a sharing time. Allow the students some time of individual work to decide on their answers then have them share their strategies in groups of 3 or 4. The question did not actually ask for the solutions to the problems but in the discussion about the strategies the solutions will arise. Following a time of group discussion, the key approaches can be discussed with the whole class with the students also suggesting how to decide on the best method to use each time. The other key point that will arise is that there is not one best method. While the balance method always works and is often the taught algorithm, it is not the most efficient method for an equation like $20/(2x + 3) = 4$ or $32/(3x + 1) = 4$. The focus of this question was solution of equations. Activity 2 is also focusing on solving equations.

Activity 2

Write down five different equations that have a solution of $x = 3.5$.

The approaches that students use to do this task can be shared with the class. In order to elicit a variety of answers from the students, criteria can be added such as at least one of the equations has to have an x on each side of the equals sign.

The activities used can be from any aspect of algebra. The critical aspect is that they are fairly open and encourage the students to share and discuss meaning. Activity 3 is an open task that focusses on equivalent expressions and raises the whole issue of simplification.

Activity 3

Ask the students to write down three different expressions equivalent to $2x + 3$. Collect verbal answers from all students (the teacher acting just as scribe), arranging them in up to five different groups on the board as students give their answers to you. The answers should be recorded on the board with no corrections. It is up to the students to discuss any discrepancies. The groups might be

- those which change the order of terms or insert symbols: e.g., $3 + 2x$, $x \times 2 + 3$;
- those in which the number term is changed: e.g., $2x + 6 – 3$, $2x + 1 + 10/5$;
- those in which the coefficient of x is altered or a series of x terms are added or subtracted: e.g., $8x/4 + 3$; $2x + 3 + x$;
- those which are a combination of the last two groups;
- a miscellaneous group which may include changes to the x: e.g., $x^2 + x + 3$.

If too many answers are coming in for any of the first three groups, ask them to try to change some other aspect of the expression. The students also will need to check that they agree with each recorded expression. When answers have been collected from the whole class, the students can explain why you have grouped them in the way you have by explaining the common aspects of each group and the differences between them. Of course with older students expressions can be with different powers. There should be class discussion about how we know the expressions are equivalent and students should try to explain how they arrived at their answers. Another way to do this is to put up the expression and focus the nature of the student answers by specific questions while still leaving them partly open. For example:

- Write down an expression with no 3 in it.
- Write down an expression with no 2 in it.
- Write down an expression with a – sign.
- Write down an expression that begins with a negative number.
- Write down an expression with a fraction in it.
- Write down an expression with a b in it.

One of the early rules students suggest is often to change the order so the negative raises that question. Students often think $a - b$ is the same as $b - a$. Rather than immediately correcting the students who suggest that the order does not matter, follow up by using the same task but with the starting point $2x - 6$, or some other similar expression. As part of the discussion one of the questions becomes, ‘How do you know when two expressions are equivalent?’ Another key issue to raise in the discussion is which of the expressions is simplest. For many students $x + x + 1 + 1 + 1$ is the simplest as it shows the basic meaning.

Activity 4

Write down an ordered pair which satisfies the equation $2x + 3y = 6$.

An important part of all these activities is the discussion which ensues. Students should explain how they arrived at their answers and discuss the relative ease of using different types of numbers and approaches.

Try it again with $y = x^2 + 3$. Did strategies change for this problem and if so why?

Concluding comments

These activities and the associated discussions are an attempt to engender in students a sense of algebra. Estimation and number sense are acknowledged as critical to our teaching. An important part of the introduction of ordinary calculators in schools is the corresponding emphasis on estimation skills as students develop the number sense necessary in tandem with calculator skills. Symbolic manipulators (computer/calculator algebra systems) are to algebra as ordinary calculators are to number, although there is one important difference. Students are continually meeting number and measurement in a variety of ways in the world around them and in their out-of-school experiences. A corresponding algebraic world experience is not as accessible. Algebra provides a language, notation and procedures that enable problems from the world to be more easily and efficiently solved. The rarity of this experience in everyday life means we must be extra careful to include experiences that can support the development of algebraic estimation skills and assist in the development of algebra sense. Our approach to teaching algebra has to allow for a variety of approaches. Efficient mental methods are not always the same as written algorithms and change more with the components of the question rather than with the nature of the question. Number sense plays an important part in this. How will the corresponding algebra sense be developed? We will need to change our teaching programs to include approaches which will build algebra sense.

References

Carpenter, T. P. & Levi, L. (2000). *Developing Conceptions of Algebraic Reasoning in the Primary Grades.* Wisconsin Center for Educational Research. Accessed 8 September 2004 from http://www.wcer.wisc.edu/ncisla/publications/reports/RR-002.pdf.

Chalouh, L. & Herscovics, N. (1988). Teaching algebraic expressions in a meaningful way. In A. F. Coxford & A. P. Shulte (Eds), *The Ideas of Algebra K–12* (pp. 33–42). Reston, Virginia: NCTM.

Collis, K. (1975). *The Development of Formal Reasoning.* Newcastle, Australia: University of Newcastle.

Filloy, E. & Sutherland, R. (1996). Designing curricula for teaching and learning algebra. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds), *International Handbook of Mathematics Education, Vol. 1* (pp. 139–160). Dordrecht, Holland: Kluwer Academic.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures.* Dordrecht, Holland: Reidel Publication.

Horne, M. & Maurer, A. (1998). A new angle on algebra. In J. Mousley & J. Gough (Eds), *Exploring All Angles* (pp. 194–200). Brunswick: Mathematical Association of Victoria.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Shulte (Eds), *The Ideas of Algebra K–12* (pp. 8–19). Reston, Virginia: NCTM.