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DUALITY, AMBIGUITY, AND FLEXIBILITY: A "PROCEPTUAL" VIEW OF SIMPLE ARITHMETIC

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In this paper we consider the duality between process and concept in mathematics, in particular, using the same symbolism to represent both a process (such as the addition of two numbers 3 + 2) and the product of that process (the sum 3 + 2). The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema. Symbolism that inherently represents the amalgam of process/concept ambiguity we call a "procept." We hypothesize that the successful mathematical thinker uses a mental structure that is manifest in the ability to think proceptually. We give empirical evidence from simple arithmetic to support the hypothesis that there is a qualitatively different kind of mathematical thought displayed by the more able thinker compared to that of the less able one. The less able are doing a more difficult form of mathematics, which eventually causes a divergence in performance between them and their more successful peers.

INTRODUCTION

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, "134 divided by 29" meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, *a/b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation.
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> —*William P. Thurston, Fields Medallist, 1990*

Mathematics has been notorious over the centuries for the fact that so many of the population fail to understand what a small minority regard as being almost trivially simple. In this article we look at the way in which mathematical ideas are developed by learners and come to the conclusion that the reason why some succeed and a great many fail lies in the fact that the more able are doing qualitatively different mathematics from the less able. The mathematics of the more able is conceived in such a way as to be—for them—relatively simple, whereas the less able are doing a different kind of mathematics that is often intolerably hard. "A small variation in notation" will be seen to hide a huge gulf in thinking between those who succeed and those who eventually fail.

PROCESS AND PROCEDURE

It will prove fruitful in our discussion to distinguish between our use of the terms *process* and *procedure*. The term *process* will be used in a general sense, as in the "process of addition," the "process of multiplication," and the "process of solving an equation," to mean the cognitive representation of a mathematical operation. It need not be a process that is currently being carried out in thought; for instance, we may speak of the process of addition without actually performing it. Nor is there any implication that the process must be carried out in a unique manner (e.g., the process of addition may be carried out by counting, by subitizing, by deduction from known facts, or by some other method). Flexibility in carrying out a process will play a fundamental role in our theory. We will use the term *procedure* in the sense of Davis (1983); it is a specific algorithm for implementing a process. For example, we see "count-on" as a procedure used to carry out the process of addition, which may be spontaneously constructed and "invented" by children (Baroody & Ginsburg, 1986), "personalised" (Gray, 1991), or taught (Fuson & Fuson, 1992).

THE PERCEIVED DICHOTOMY BETWEEN PROCEDURE AND CONCEPT

Hardly a decade passes without concern being expressed over the general level of children's attainment in mathematics, the quality of their learning, or the nature of the mathematics curriculum. In the U.S. the NCTM *Standards* (1989) reflect the perceived need to improve children's performance. Within the United Kingdom the imposition of a National Curriculum (1989) is aimed at raising standards of performance in all subjects, including mathematics. The requirements of this curriculum distinguish between the skills or procedures that individuals need to have acquired in order to do things, and the concepts or basic facts, which they are expected to know, on which they operate with their skills. This suggests a fundamental dichotomy between procedures and concepts, between things to do and things to know. However, in mathematics, we shall see that the truth is somewhat different.

Procedural aspects of mathematics focus on routine manipulation of objects that are represented either by concrete materials, spoken words, written symbols, or mental images. It is relatively easy to see if such procedures are carried out adequately, and performance in similar tasks is often taken as a measure of attainment in these skills.

Conceptual knowledge, however, is harder to assess. It is knowledge that is rich in relationships. Hiebert and Lefevre (1986) describe conceptual knowledge as

> a connected web ... a network in which the linking relationships are as prominent as the discrete pieces of information.... A unit of conceptual knowledge cannot be an isolated piece of information; by definition it is part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information. (pp. 3–4)

Flexible thinking using conceptual knowledge is likely to be very different from thinking based on inflexible procedures. Yet procedures still form a basic part of mathematical development. Piaget (1985, p. 49) speaks of the way in which "actions or operations become thematized objects of thought or assimilation." What is important is the cognitive shift from mathematical processes into manipulable mental objects.

PROCESS BECOMING CONCEIVED AS CONCEPT

The notion of actions or processes becoming conceived as mental objects has featured continually in the literature. Dienes (1960) uses a grammatical metaphor to describe how a predicate (or action) becomes the subject of a further predicate, which may in turn become the subject of another. He claims that

> people who are good at taming predicates and reducing them to a state of subjection are good mathematicians. (p. 21)

In an analogous way, Greeno (1983) defines a "conceptual entity" as a cognitive object that can be manipulated as the input to a mental procedure. The cognitive process of forming a (static) conceptual entity from a (dynamic) process has variously been called "entification" (Kaput, 1982), "reification" (Sfard, 1989, 1991), and "encapsulation" (Dubinsky, 1991). We shall use these terms interchangeably in the remainder of the article, favoring the word encapsulation.

Encapsulation is seen as operating on successively higher levels (Piaget, 1972) so that

> the whole of mathematics may therefore be thought of in terms of the construction of structures, ... mathematical entities move from one level to another; an operation on such "entities" becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by "stronger" structures. (p. 70)

From the viewpoint of a professional mathematician (Thurston, 1990):

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (p. 847)

Sfard (1991) expresses the way in which the stratification occurs by talking about operational mathematics, in which the operations at one level become reified as objects to become basic units of a higher level theory. We now provide a perspective on the encapsulation of process as object through an analysis of simple arithmetic.

At the foundation of arithmetic is the concept of number and, we suggest, the ability to count. There are alternative views (Wohlwill & Lowe, 1962;

Piaget, 1952), and we are aware of the different views that attach to children's use of quite small numbers that can be relatively easily subitized and directly perceived as properties of a physical collection (Klahr & Wallace, 1976; Gelman & Gallistel, 1986). However, more recent analysis of the development of number concepts indicates that counting plays a sophisticated and central role (Wagner & Walters, 1982; Fuson & Hall, 1983; Gelman & Meck, 1986). The sequence of number words becomes part of a procedure to point at successive elements; each number word is uttered in turn until the last word is identified as the number of elements in the collection. In this manner we see the process of counting encapsulated as the concept of number.

This relationship between the process of counting and the concept of number allows us to reconsider the apparent dichotomy between procedural and conceptual knowledge and to consider how this relates to the divergence between inflexible procedures and flexible concepts.

THE ROLE OF SYMBOLS

The manner in which symbols are used will play a pivotal role in our discussion of the relationship between process and concept. For this purpose we regard a symbol as something that is perceived by the senses. It can be written or spoken so that it can be seen or heard. What is important about the physical representation for our theoretical perspective is the way in which it is interpreted by different individuals or by the same individual at different times. In particular, we will be interested in the way in which a symbol can be conceived as representing a process or an object.

It is interesting to note that Sinclair and Sinclair (1986) sense that with preschool children—for whom written symbolism has yet to be developed—the distinction between procedural and conceptual knowledge seems far less appropriate. Following Piaget, their discussion focuses once more on the theme of action (process) becoming the object of thinking, the process becoming the concept.

Sfard (1989) comments that although processes and objects are ostensibly incompatible, they are in fact compatible and can be simultaneously conceived of as mathematical notions. Yet she asks, "How can anything be a process and an object at the same time?" (p. 151)

We suggest that the answer lies in the way that professional mathematicians cope with this problem. They employ the simple device of using the same notation to represent both a process and the product of that process. As Thurston's father noted in the initial quotation, *a/b* and *a* divided by *b* are just synonyms—a small variation in notation. In practice there is rarely a variation; the same notation is used for either a process or the product of that process.

THE AMBIGUITY OF SYMBOLISM FOR PROCESS AND CONCEPT

The ambiguous use of symbols for process or product pervades the whole of mathematics:

- The symbol $5 + 4$ represents both the process of adding by counting all or counting on and the concept of sum ($5 + 4$ is 9).
- The symbol 4×3 can stand for the process of repeated addition, "four threes," that must be carried out to obtain the product of four and three, which is the number 12.
- The symbol 3/4 stands for both the process of division and the concept of fraction.
- The symbol +4 stands for both the process of "add four," or shift four units along the number line, and the concept of the positive number +4 (initially many educators use different notations such as $^{+}4$ for the positive number and +4 for "add four," but these are usually suppressed at a later stage).
- The symbol −7 can stand for both the process of "subtract seven," or shift seven units in the opposite direction along the number line, and the concept of the negative number −7 (again, initially noted as $^{-}7$).
- The algebraic symbol $3x + 2$ stands both for the process "add three times x and two" and for the product of that process, the expression "$3x + 2$."
- The trigonometric ratio sine = $\frac{\text{opposite}}{\text{hypotenuse}}$ represents both the process for calculating the sine of an angle and its value.
- The function notation $f(x) = x^2 - 3$ simultaneously tells both how to calculate the value of the function for a particular value of x and encapsulates the complete concept of the function for a general value of x.
- An "infinite" decimal representation $\pi = 3.14159\ldots$ is both a process of approximating π by calculating ever more decimal places and the specific numerical limit of that process.
- The notation $\lim_{x \to a} f(x)$ represents both the process of tending to a limit and the concept of the value of the limit, as does

$$\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} \sum_{k=1}^{n} a_k,\ \text{and}\ \lim_{\delta x\to 0} \sum_{x=a}^{b} f(x)\delta x.$$

Mathematicians abhor ambiguity and so they rarely speak of it, yet ambiguity is widely used throughout mathematics. We believe that the ambiguity in interpreting symbolism in this flexible way is at the root of successful mathematical thinking. We further hypothesize that its absence leads to stultifying uses of procedures that need to be remembered as separate devices in their own context ("do multiplication before addition," "turn upside down and multiply," "two negatives make a positive," "add the same thing to both sides," "change sides, change signs," "cross multiply," etc.)

We conjecture that the dual use of notation as process and concept enables the more able to "tame the processes of mathematics into a state of subjection"; instead of having to cope consciously with the duality of concept and process, the good mathematician thinks ambiguously about the symbolism for product and process. We contend that the mathematician simplifies matters by replacing the cognitive complexity of process-concept duality by the notational convenience of process-product ambiguity.

THE NOTION OF PROCEPT

We do not consider that the ambiguity of a symbolism expressing the flexible duality of process and concept can be fully utilized if the distinction between process and concept is maintained at all times. It is essential that we furnish the cognitive combination of process and concept with its own terminology. We therefore use the portmanteau word "procept" to refer to this amalgam of concept and process represented by the same symbol. However, we wish to do this in a way that reflects the cognitive reality.

We propose the following preliminary definition: An *elementary procept* is the amalgam of three components: a *process* that produces a mathematical *object*, and a *symbol* that represents either the process or the object.

This definition allows the symbolism to evoke either process or concept, so that a symbol such as $2 + 3$ can be seen to evoke either the process of addition of the two numbers or the concept of sum.

The definition caused us a great deal of heart searching, because we wanted it to reflect the observed cognitive reality. In particular, we wanted to encompass the growing compressibility of knowledge characteristic of successful mathematicians. Here, not only is a single symbol viewed in a flexible way, but when the same object can be represented symbolically in different ways, these are often seen not only as different processes to give the same object but as different names for the same object.

In order to reflect this growing flexibility of the notion and the versatility of the thinking processes we extend the definition as follows: A *procept* consists of a collection of elementary procepts that have the same object.

In this sense we can talk about the procept 6. It includes the process of counting 6 and a collection of other representations such as $3 + 3$, $4 + 2$, $2 + 4$, 2×3, $8 - 2$, and so on. All of these symbols may be considered to represent the same object, yet indicate the flexible way in which 6 may be decomposed and recomposed using different processes.

We are well aware that mathematically we could put an equivalence relation on elementary procepts, to say that two are equivalent if they have the same object and then define a procept to be an equivalence class of elementary procepts. However, we feel that this kind of mathematical precision overcomplicates the cognitive reality. The nature of the procept is dependent on the cognitive growth of the child. It starts out with a simple structure and grows in interiority in the sense of Skemp (1979). We see an elementary procept as the first stage in the dynamic growth of a procept, rather than an element in an equivalence class.

We see number initially as an elementary procept. A symbol such as 3 evokes both the counting process "one, two, three" and the number itself. The word *three* (and its accompanying symbol 3) can be spoken, it can be heard, it can be written, and it can be read. These forms of communication allow the symbol to be shared in such a way that it has, or seems to have, its own shared reality. Three is an abstract concept, but through using it in communication and acting upon it with the operations of arithmetic, it takes on a role as real as any physical object.

The symbol 3 grows in richness of meaning, inextricably linking both procedural and conceptual aspects. It includes the procedural aspects of counting and the conceptual relationships in which the same object is represented by different symbols; $1 + 1 + 1$, $2 + 1$, $1 + 2$, and $4 - 1$ all have an output 3 and together form part of the procept 3. They allow the number 3 to be decomposed and recomposed in a variety of ways reflecting the different processes available to produce the same object.

In this way the various different forms combine to give a rich conceptual structure in which the symbol 3 expresses all these links, the conceptual ones and the procedural ones, the processes and the product of those processes. The combination of conceptual and procedural thinking in this manner we term *proceptual thinking*.

THE GROWTH OF PROCEPTUAL THINKING IN ARITHMETIC

The procedural and conceptual approaches that children use to form the sum of two or more amounts introduced through word problems have been well documented (e.g., Fuson, 1982; Carpenter, Hiebert, & Moser, 1981, 1982). Translating some or all of these approaches into a conceptual hierarchy for addition formed part of the focus of these and other studies (Herscovics & Bergeron, 1983; Secada, Fuson, & Hall, 1983; Gray, 1991; Fuson & Fuson, 1992).

These approaches for finding a sum involve a number of different procedures, including "count-all" (count each set separately then count the two together), "count-on from first" (count-on the number of elements in the second set, starting from the number in the first set), "count-on from largest" (put the larger set first and count-on the smaller number of elements in the second set), together with higher-order strategies, such as "knowing the fact" or "deriving new facts from known facts." There are corresponding procedures for subtraction: "take-away" (count the big set, count the subset to be taken away, then count the set that remains), "count-back" (start from the larger number and count-back down the number sequence to find the number remaining), "count-up" (start from the number to be taken away and count-up to the number given), together with higher-order strategies, either "knowing the facts" or "deriving" the facts.

Even finer gradations of these categories have been proposed and can be helpful in distinguishing children's thinking processes. However, here we wish to use the notion of procept to conceptualize the cognitive development in an integrated manner, referring only to the growing facility for compression of ideas from procedures of counting to the procept of number.

Using the summary given by Carpenter et al. (1981) and resorting to reanalysis of the evidence given by Gray (1991) we specify that the procedure of count-all consists of three separate subprocedures: count the first set, count the second set, then combine the sets as a single set and count all the objects (Figure 1).

We conjecture that the most salient memory that the child has of this process is the final object counted. This represents the value of the set that is the union of two sets formed from the two subprocedures, which involved counting two and counting three. The total of this set, five, is the last point of reference for the child. Because such a procedure occurs in time, it is hypothesized that any proceptual relationship between the input (3 plus 2) and the output (5) is likely to be obscured by the lengthy counting routine used to obtain the solution. The nature of such a procedure can mitigate against the encapsulation of 3 + 2 = 5 as a known fact. We suggest, then, that count-all is a procedure that extends the counting process and is unlikely to lead directly to an encapsulated procept.

The count-on procedure is a more sophisticated strategy than count-all (e.g., Secada, et al., 1983; Carpenter, 1986; Baroody & Ginsburg, 1986; Gray, 1991). The notion of elementary procepts helps our analysis of the procedure. To one number a second is added through a count-on procedure. (It is actually a sophisticated double counting procedure where 3 + 2 involves saying "four,

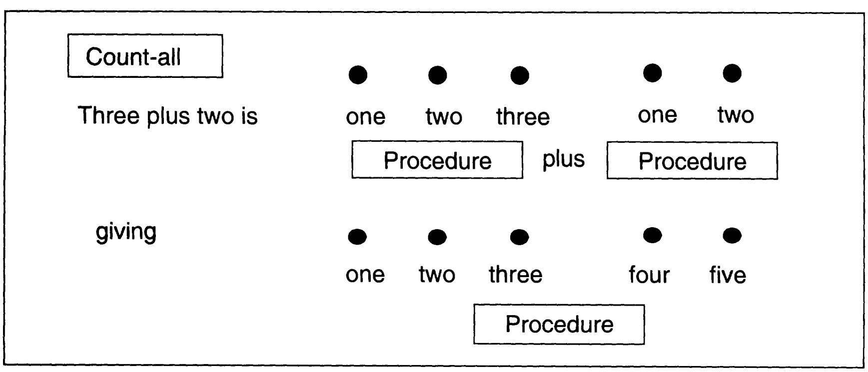


Figure 1. Count-all as a combination of procedures.

five" while simultaneously keeping track that "two" extra numbers are being counted.) We therefore see count-on as "elementary procept plus procedure": one number is incremented in ones to form a successive series of elementary procepts through a counting procedure (Figure 2).

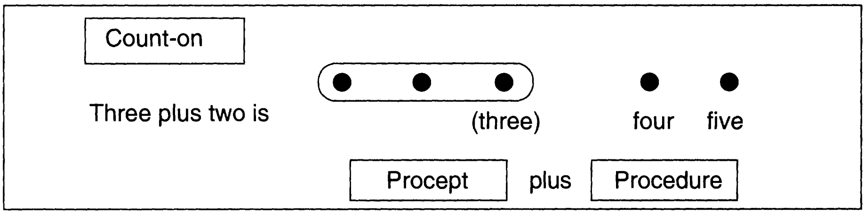


Figure 2. Counting-on as procept plus procedure.

We believe that count-on as a procedure can have two qualitatively different outcomes, as a (counting) procedure of addition or as the procept of sum.

1. Count-on as procedure is essentially a compression of count-all into a shorter procedure. It remains a procedure that takes place in time so that the child is able to compute the result without necessarily linking input and output in a form that will be remembered as a new fact. Some children—often with a limited array of known facts—may become so efficient in counting that they use it as a universal method that does not involve them in the risk of attempting to use a limited number of known facts (see also Steinberg, 1985).

2. Count-on leading to procept produces a result that is seen both as a counting procedure and a number concept. The notation 3 + 2 is seen to represent both the process of addition and the result of that process, the sum.

When input numbers and their sum can be held in the mind simultaneously then the result is a meaningful, known fact that may be envisioned as a flexible combination of procept and procept to give a procept (Figure 3).

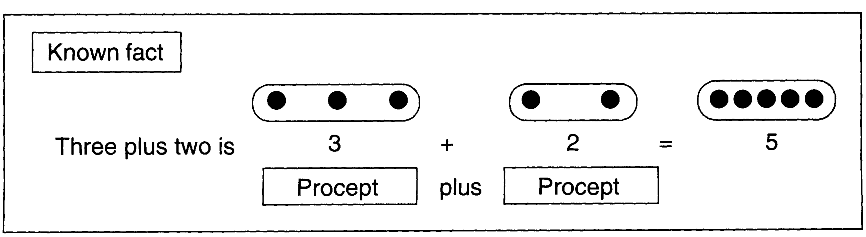


Figure 3. (Meaningful) known fact as procept plus procept.

It is important here to distinguish between a meaningful known fact generated by this flexible form of thinking and a fact that is remembered by rote. In any isolated incident such a distinction may be hard to make. The difference is more apparent when such facts are decomposed and recomposed to give derived facts. Merely knowing facts does not necessarily lead to deriving facts, as we shall see shortly. But the language used by children who do derive facts shows that they freely decompose and recompose the component parts in a proceptual way. For instance, faced with "four and five," one may know that "four and four makes eight," and respond that it is "one more," which is "nine." Some facts such as, for example, "16 + 3 is 19" based on "6 + 3 is 9," can be derived so fast that the process is virtually instantaneous. On occasion it may be difficult to distinguish between a known fact and a quickly constructed derived fact.

The need for flexibility in arithmetic is a regular feature in the literature. For instance, Fuson, Richards, and Briars (1982) and Steffe, von Glasersfeld, Richards, and Cobb (1983) suggest that the use of a sequence of number words for the solution of addition and subtraction problems leads to the understanding that addition and subtraction are inverse operations, and this contributes to the flexibility of solving addition and subtraction problems. However, proceptual flexibility gives new insight. The existence of flexible proceptual knowledge means not only that the number 5 can be seen as 3 + 2 or 2 + 3 but that if 3 and something makes 5, then the something must be 2. In proceptual thinking, addition and subtraction are so closely linked that subtraction is simply a flexible reorganization of addition facts.

In proceptual thinking, addition as count-on is considered to have subtraction as its inverse through count-back or count-up. We shall see that less successful children often favor count-back as the natural reverse process even though its cognitive complexity is enormous. The child must count the number sequence in reverse starting from the larger number and keep track simultaneously of how many numbers have been counted. To obtain the solution to an arithmetical problem such as 16 – 13 = ___ by count-back requires the recitation of 13 numbers in reverse order from 16 down. Such procedures, especially when carried out by less successful children, often result in error. Because the proceptual thinker has a simpler task than the procedural thinker, the likely divergence between success and failure is widened.

The fundamentally different ways of thinking exhibited by children performing arithmetic, usually represented by the terms procedural and conceptual, may be described more incisively as procedural and proceptual. Proceptual thinking includes the use of procedures. However, it also includes the flexible facility to view symbolism either as a trigger for carrying out a procedure or as the representation of a mental object that may be decomposed, recomposed, and manipulated at a higher level. This ambiguous use of symbolism is at the root of powerful mathematical thinking and makes it possible to overcome the limited capacity of short-term memory. It enables a symbol to be maintained in short-term memory in a compact form for mental manipulation or to trigger a sequence of actions in time to carry out a mathematical process. It includes both concepts to know and processes to do.

QUALITATIVELY DIFFERENT APPROACHES TO SIMPLE ARITHMETIC

The evidence of qualitatively different approaches that can be interpreted using the notion of procept arises from data collected by Gray (1991). He interviewed a cross-section of children aged 7 to 12 from two mixed-ability English schools to discern their methods of carrying out simple arithmetic exercises. Toward the end of the school year, when the teachers had developed intimate knowledge of the children for over 6 months, he asked the teachers of each class to divide their children into three groups—"above average," "average," and "below average" according to their performance of arithmetic—and to select two children from each group who were "representative" of each group. The two schools each provided 6 children from each of six year groups, making 72 children in all, 12 from each year divided into three groups of 4 children according to their teachers' perceptions of their performance in arithmetic. In what follows we shall refer to the year groups by age, so that, for instance, 9+ refers to children who would be nine during the school year. They were interviewed over a 2-month period starting 6 months after the beginning of the year, so at the time of interview a child designated as 9+ would be in the range 8 years 6 months to 9 years 8 months.

Figures 4 and 5 consider the types of response made by the above-average and the below-average children to a range of addition and subtraction problems subdivided into three levels. Figure 4 (adapted from Gray & Tall, 1991) illustrates the responses to the three categories of addition problems considered:

A: Single-digit addition with a sum of 10 or less (e.g., 6 + 3, 3 + 5)

B: Addition of a single-digit number to a teen number, the sum being 20 or less (e.g., 18 + 2, 13 + 5)

C: Addition of two single-digit numbers with a sum between 11 and 20 (e.g., 4 + 7, 9 + 8)

(In the figure each of these three categories is represented for the three ability groups at each age. For instance, of children aged 8+ in category B, those above average obtained 30% of the solutions through known facts, 61% through derived facts, and 9% through count-on. In contrast, the below-average children of the same age obtained 6% of category B solutions through known fact, 72% through count-on, and 22% through count-all.)

The striking difference between the two groups is seen by a comparison of the use of procedural methods (counting) and the use of derived facts.

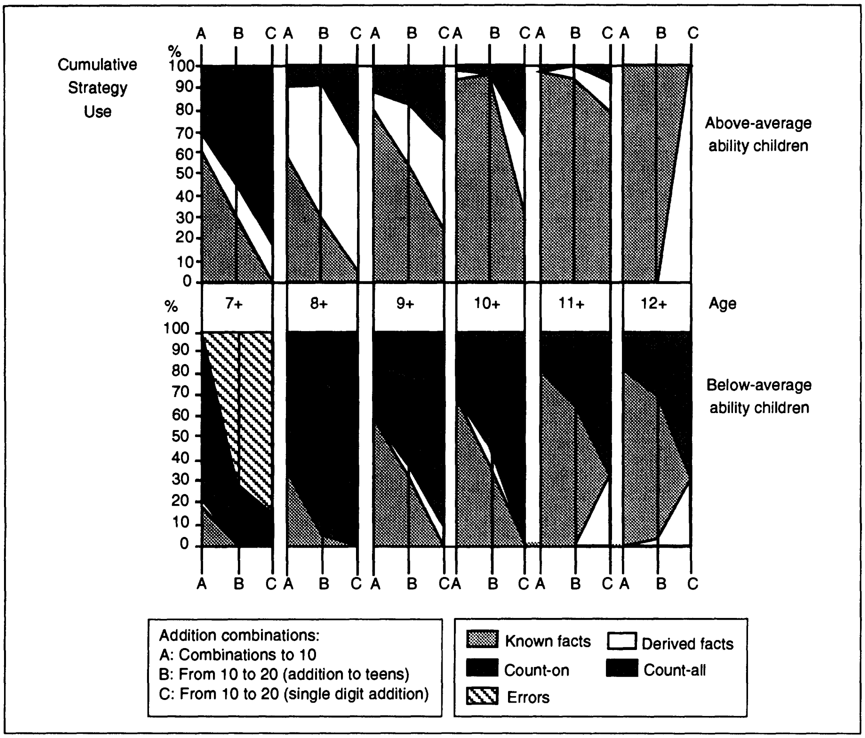


Figure 4. Strategies used to obtain solutions to simple addition combinations by groups of below-average and above-average children of different ages.

Note that when count-all is used fairly extensively, then derived facts hardly occur. Apart from three exceptions—a single instance at 7+ and two at 10+, all of whom derived the solution to 4 + 5,—no below-average child provides any further evidence of the use of derived facts to obtain solutions to category A (single-digit problems less than 10). Indeed, even the derived solution to 4 + 5 was almost procedural: "I always do this one by adding 4 and 4 and 1 more." The above-average children make pertinent use of derived facts in category A, particularly at age 8+, where there is an explosion in their use within every category.

In most cases the 7+ below-average children were unable to use any appropriate method to obtain correct solutions for the category B and C problems. Older children invariably used counting techniques to solve category B and C problems, even when they knew the answer to the corresponding category A problem. (For instance, they might know 4 + 4, yet count 14 + 4.)

In sharp contrast, few above-average children counted category B problems. Faced with a problem such as 15 + 4, they either knew 5 + 4 is 9 and

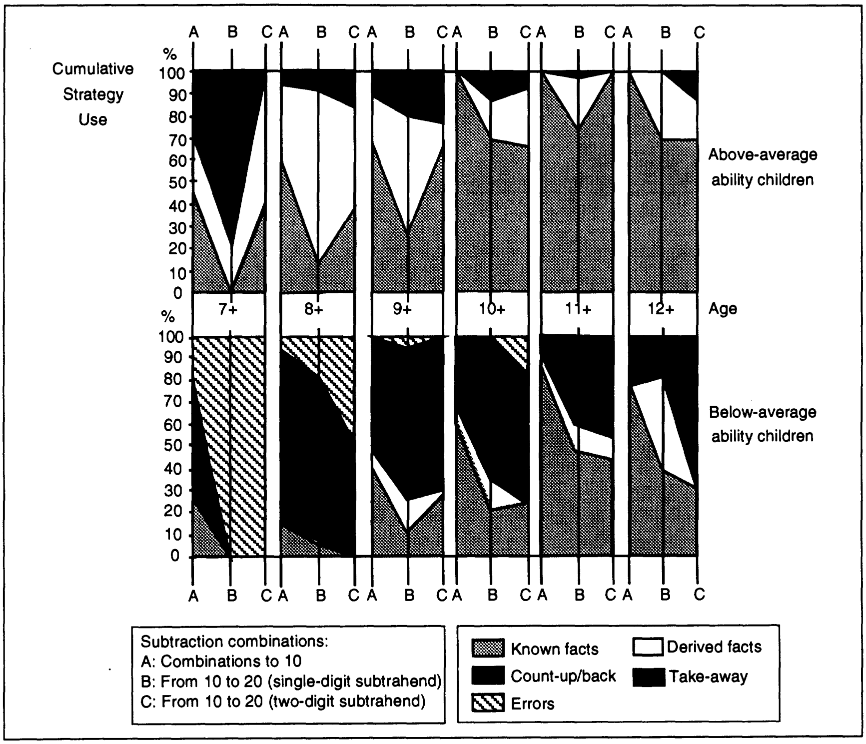


Figure 5. Strategies used to obtain solutions to simple subtraction combinations by groups of below-average and above-average children of different ages.

added 10 to get 19, or they derived the value of 5 + 4 by, for instance, knowing "two fives are ten" so 5 + 4 is 9 and adding 10. The below-average children remembered almost no category C combinations, the single exception being 9 + 8, which several below-average students aged 11+ and 12+ derived by knowing 9 + 9.

Figure 5 (adapted from Gray & Tall, 1991) concentrates on three categories of subtraction:

A: Single-digit subtraction (e.g., 8 – 2)

B: Subtraction of a single-digit number from a number between 10 and 20 (e.g., 16 – 3, 15 – 9)

C: Subtraction of one two-digit number between 10 and 20 from another (e.g., 16 – 10, 19 – 17)

Once again categories including take-away (the subtraction equivalent of count-all) have few instances of derived facts. The apparent exception among the above-average 7-year-olds is provided by one child who gave all the examples of take-away in category A. Where he needed to use procedural methods in the other categories this child used count-back for combinations such as 13 – 2 and 16 – 3 but count-up for those such as 15 – 9 and 17 – 13. Note the low incidence of known facts among the 7+ and 8+ below-average children and the related absence of derived facts; wherever derived facts are used by these children in category A it is once again the near double 9 – 5 that provides the stimulus. Meanwhile the 8+ above-average children once again have over 50% known facts in category A and a high incidence of derived facts in all three categories. With a good knowledge of subtraction facts to 10, these more successful children are able to derive almost everything they do not know and only occasionally resort to counting.

We see that the 10+ below-average group, sometimes called slow learners in the U.K., appear to possess the same profile of known facts as the 8+ above-average group, but they do not use these facts in the same way. The above-average 8+ children derive most facts that they do not know; the below-average 10+ children derive no facts in category C and only occasional facts in categories A and B. Instead, they count. We suggest that the phrase "slow learners" is therefore a misnomer. The less able do not simply learn the same techniques more slowly. They develop different techniques. Throughout the age range the above-average show a high incidence of known facts, and nearly everything they do not know they derive. The below-average students rarely use derived facts; instead, they almost always count.

We conjecture that the differences in behavior between the above- and below-average groups is caused by what we term the proceptual divide. Proceptual thinking includes the meaningful use of known facts to arrive at solutions through derived facts. It may also include the use of a procedure. A single-item analysis of children's responses may not be sufficient to allow a distinction to be made—how do we distinguish between a rote learned fact and a meaningful known fact? If a child uses a procedure to solve one problem, does that mean that the next problem will also be solved procedurally? Only through analysis of the solution strategies to a range of problems may we get our answer. We may then see either the flexibility, which is a keynote of the proceptual interplay between conceptual and procedural methods, or the limitations imposed by the reliance on fixed counting procedures. The latter often provide considerable success at one level but may ultimately lack the generality to lead to success in more sophisticated problems.

Careful reconsideration of the individual data shows that not only do the less able count more, there is even a difference between the two groups in how they count. The below-average group nearly always selected count-back as the natural procedure for take-away, so that 19 – 17 is likely to be calculated by laboriously counting back 17 from 19, a procedure that usually ends in failure (particularly among the younger children). When the above-average children counted, they were more likely to select the more appropriate strategy, in this case counting up from 17 to 19.

Empirical evidence therefore supports the hypothesis that the more able tend to display more flexible proceptual techniques (including the selection of more appropriate procedures), whereas the less able rely on less flexible procedural methods of counting.

INDIVIDUAL CASES

Individual examples make this more apparent. Michael (9+) is categorized as below average by his teacher. He chose to write 18 – 9 in the standard vertical layout and, as is usual in the decomposition process, put a little 1 by the 8 like this:

$$\begin{array}{r} 1\,{}^{1}8 \\ -\quad 9 \\ \hline \end{array}$$

"This is the easy way of working it out. I can't take 9 from 8 but if I put a little 1 it makes it easier because now it's 9 from 18." He failed to realize that this is the same sum he started with, and after a considerable time trying to cope with this problem, he resorted to his more usual procedure for subtraction by placing 18 marks from left to right on his paper, then starting from the left and counting from one to nine as he crossed out 9 marks. He recounted the remaining marks from left to right to complete the correct solution by take-away.

The less-able children are often placed in difficulties as they grow older because they feel pressure to conform and not use "baby" methods of counting.

Jay (10+) solved the problem 5 – 4 by casually displaying five fingers on the edge of the desk and counted back, "five, ... four, three, two, one." At each count apart from the five he put slight pressure on each finger in sequence. The solution was provided by the last count in the sequence. When attempting 15 – 4 he wanted to use a similar method but had a problem, declaring, "I'm too old for counters!" but neither did he want to be seen using his fingers, because "my class don't use counters or fingers." He felt he should operate in the same way as other children in his class (most of whom appeared to recall the basic facts from memory), yet he did not know the solutions and knew that he required a counting support. His use of fingers for obtaining solutions for number combinations was almost always covert. When dealing with combinations to 20 he combined a casual display of 10 splayed fingers on the edge of his desk with an imagined repetition of his fingers just off the desk. He spent a considerable time obtaining individual solutions and had a tendency to be very cautious in giving responses. He used his imaginary fingers to attempt to find a solution to 15 – 9 by counting back. Eventually he became confused and couldn't complete the problem.

Some of those deemed below average have been classified as using some derived facts. However, this may be accompanied by a display of fingers for visual support, an action never performed by the above-average students. This was categorized as derived fact because there was no visible evidence of actual counting. The fingers needed to be held up for the number facts to be recalled from the finger layout.

For instance, Karen (11+), the most successful of the below-average students, made considerable use of her fingers in an idiosyncratic inventive manner. To perform the calculation 15 – 9, she held out five fingers on her left hand and closed it completely; she then held up four fingers on her right hand, closed them and opened the right thumb, then redisplayed the five fingers of her left hand at the same time, and responded, "Six." The whole procedure took about 3 seconds.

Her explanation showed a subtle understanding of number relationships (Figure 6).

Nevertheless, the tortuous route that she followed showed that her inventiveness tended to relate to individual calculations and applied only to small numbers she could represent using her hands. Other below-average children who attempted to derive facts often had to do this on the basis of a limited number of known facts that might not furnish the most efficient way to perform the calculation. For example, Michelle (aged 10+), faced with 16 – 3, said "10 from 16 leaves 6, 3 from 10 leaves 7, 3 and 7 makes 10 and another 3 is 13." Michelle seeks to find familiar number bonds to solve the problem. She sees 16 as 6 and 10, but takes the 3 from the 10 rather than from the 6 and ends up having to do the additional sum 6 and 7.

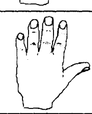
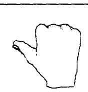
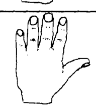
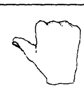
	Display		Explanation to Calculate 15–9	
	Left hand	Right hand	Child's explanation	Interviewer's comment
Stage 1			Fifteen is ten and five. Forget the ten.	Five fingers shown on left hand, (Other ten presumably held in mind.) Right hand closed.
Stage 2				The child displays the nine to be taken away as a five and a four.
Stage 3				The left hand is closed, to cancel the displayed five leaving the previously displayed four.
Stage 4			Four from one of the fives making ten leaves one.	The remaining four are taken from one of the fives held in the mind.
Stage 5			One and the other five from the ten make six.	Remaining five in mind now displayed, giving a total of 5 and 1, which is 6.

Figure 6. Subtracting 9 from 15 by an inventive route.

THE PROCEPTUAL DIVIDE

The more able children tend to display proceptual thinking, whereas the less able are more procedural. The characteristics of these two forms of thinking may be summarized as follows:

1. Procedural thinking is characterized by a focus on the procedure and the physical or quasi-physical aids that support it. The limiting aspect of such thinking is the more blinkered view that the child has of the symbolism: numbers are used only as concrete entities to be manipulated through a counting process. The emphasis on the procedure reduces the focus on the relationship between input and output, often leading to idiosyncratic extensions of the counting procedure that may not generalize.
2. Proceptual thinking is characterized by the ability to compress stages in symbol manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways.

Flexible strategies used by the more able students produce new known facts from old, giving a built-in feedback loop that acts as an autonomous knowledge generator. The least successful have only a procedure of counting, which grows ever more lengthy as the problems grow more complex. In between these extremes, the less able who do attempt to derive facts from a limited range of known facts may end up following an inventive but tortuous route that succeeds only with the greatest effort. The high sense of risk generated may then lead to such a child falling back on the security of counting. We therefore hypothesize that what might be a continuous spectrum of performance tends to become a dichotomy in which those who begin to fail are consigned to become procedural. We believe that this bifurcation of strategy—between flexible use of number as object or process and fixation on procedural counting—is one of the most significant factors in the difference between success and failure. We call it the *proceptual divide.*

It is our contention that whereas more able younger children evoke proceptual thinking to use the few combinations already known to establish more, less able children remain concerned with the procedures of counting and apply their efforts to developing competence with them. Procedural thinking in the context of developing competency with the number combinations can give guaranteed success and efficiency within a limited range of problems. But this efficiency with small numbers is unlikely to lead to success with more complex problems as the children grow older. Their persistence in emphasizing procedures leads many children inexorably into a cul-de-sac from which there is little hope of future development.

Figures 7 and 8 illustrate the divergence in strategy between the more and less able in the study. They pair age groups together, 2 years at a time to give more viable group sizes, to allow a more detailed analysis of the data in Figures 4 and 5. Combinations are arranged in order of difficulty, established by considering the overall percentage of children within the sample who responded to individual combinations through the use of known facts. The three main groupings correspond to the categories previously considered in Figures 4 and 5.

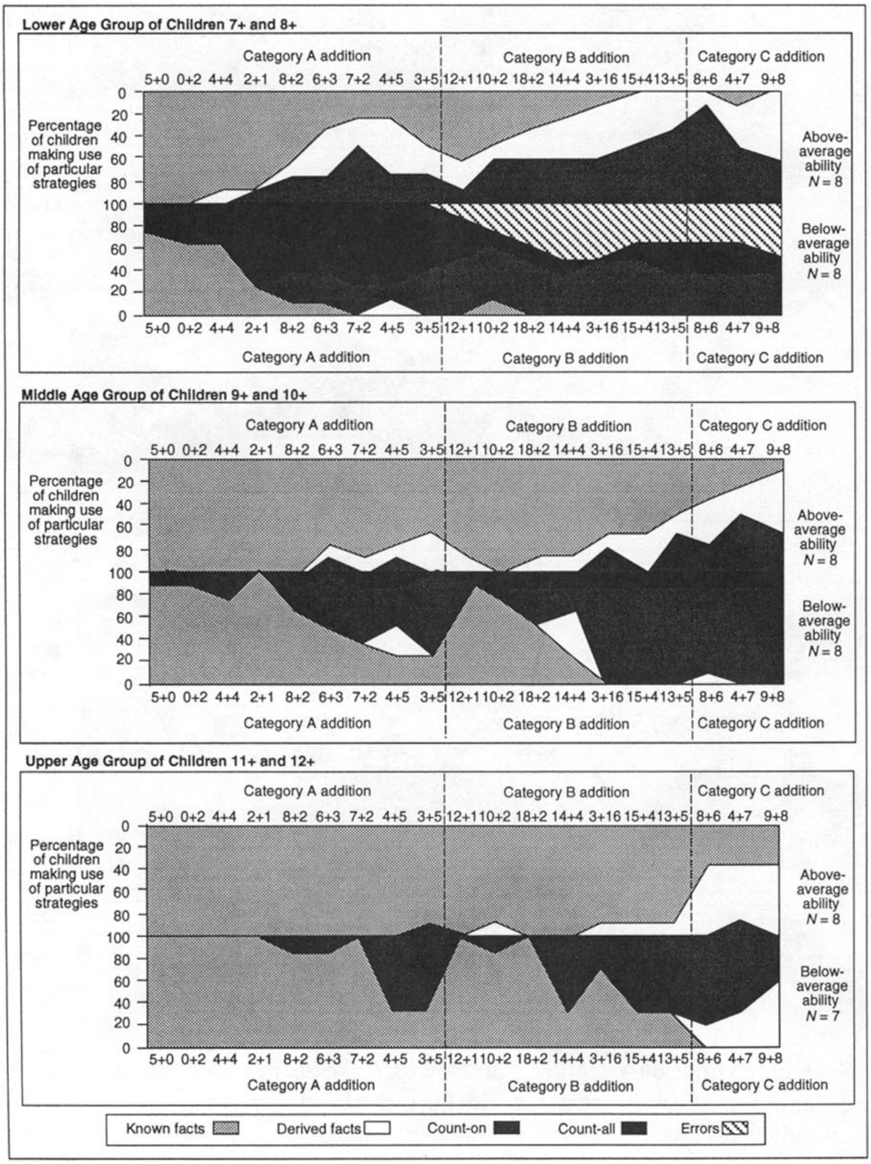


Figure 7. Diverging approaches to basic addition combinations, with age and ability comparisons.

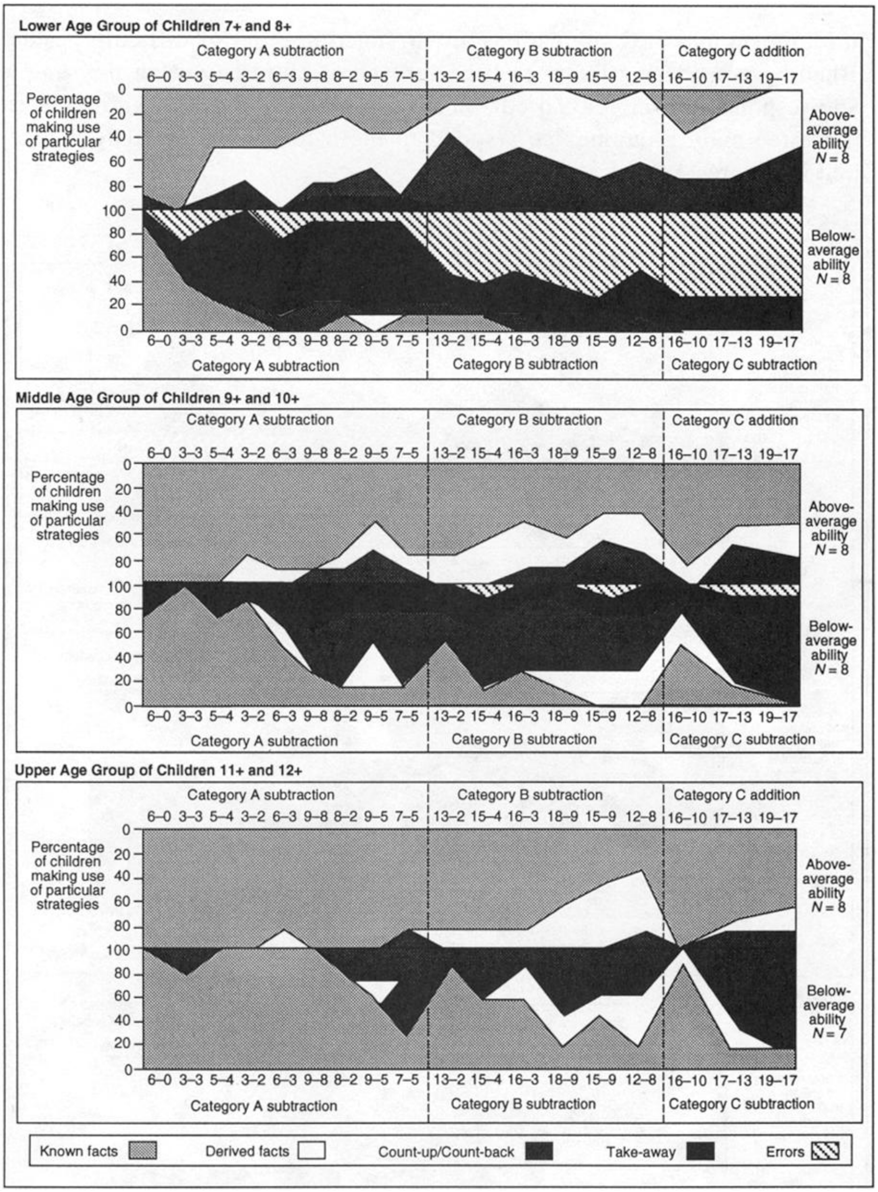


Figure 8. Diverging approaches to basic subtraction combinations, with age and ability comparisons.

The graphs not only starkly illustrate differences between the below-average and the above-average children but also how combinations evoke particular responses. Note how combinations involving single digits and a sum between 10 and 20 evoke the use of derived facts by the upper age group of 11- and 12-year-olds. Note also how the extensive use of procedural methods among the youngest below-average group to obtain solutions to number combinations to 10 fails to provide them with a means of obtaining solutions to harder problems.

See how above-average students make use of very few known facts to establish solutions through the use of derived facts. For instance, "6 – 3 is 3 because two threes are six"; "4 + 7 is 11 because 3 and 7 is 10"; "18 – 9 is 9 because 9 × 2 is 18"; "8 + 6 is 14 because two 7s are 14." Simpler facts become known facts (or perhaps instantaneous derived facts). Harder combinations are less often committed to memory, perhaps because the more able realize that it is just as efficient to derive them when required.

Note that even when below-average children know a substantial number of facts they make very little use of derived solutions. Contrast the efficient solution to 8 + 6 above with a solution derived by a less able child. Stuart (aged 10+) responded to this problem by saying, "I know 8 and 2 is 10, but I have a lot of trouble taking 2 from 6. Now 8 is 4 and 4; 6 and 4 is 10; and another 4 is 14." We may feel we should congratulate Stuart for the breadth of arithmetical manipulation that he displays, but the truth of the matter is that his particular approach indicates not so much what he knows as what he does not know. He knows number combinations that make 10 but in this context has difficulty with 6 – 2! His idiosyncratic methods of solution place a severe burden of inventiveness on him to solve arithmetic problems. It may in the long term prove too great a burden to bear.

THE CUMULATIVE EFFECT OF THE PROCEPTUAL DIVIDE

Proceptual encapsulation occurs at various stages throughout mathematics: repeated counting becoming addition, repeated addition becoming multiplication, and so on, giving what is usually considered by mathematics educators a complex hierarchy of relationships (Figure 9).

The less able child who is fixed in process can only solve problems at the next level up by coordinating sequential processes. This is, for them, an extremely difficult process. If they are faced with a problem two levels up, then the structure will almost certainly be too burdensome for them to support (see Linchevski & Sfard, 1991). Multiplication facts are almost impossible for them to coordinate while they are having difficulty with addition. Even the process of reversing addition to give subtraction is seen by them as a new process (count-back instead of count-up).

The more able, proceptual thinker is faced with an easier task. The symbols for sum and product again represent numbers. Thus counting, addition, and multiplication are operating on the same procept, which can be decomposed into process for calculation purposes whenever desired. A proceptual view that amalgamates process and concept through the use of the same notation therefore collapses the hierarchy into a single level in which arithmetic operations (processes) act on numbers (procepts) (Figure 10).

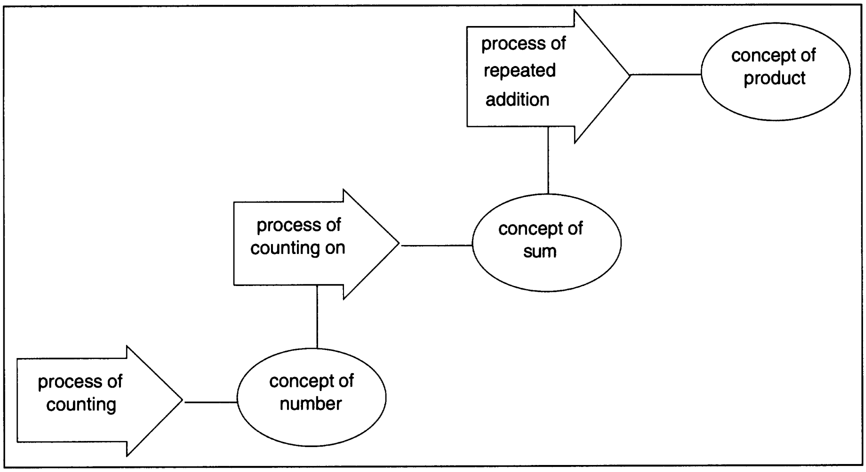


Figure 9. Higher-order encapsulations.

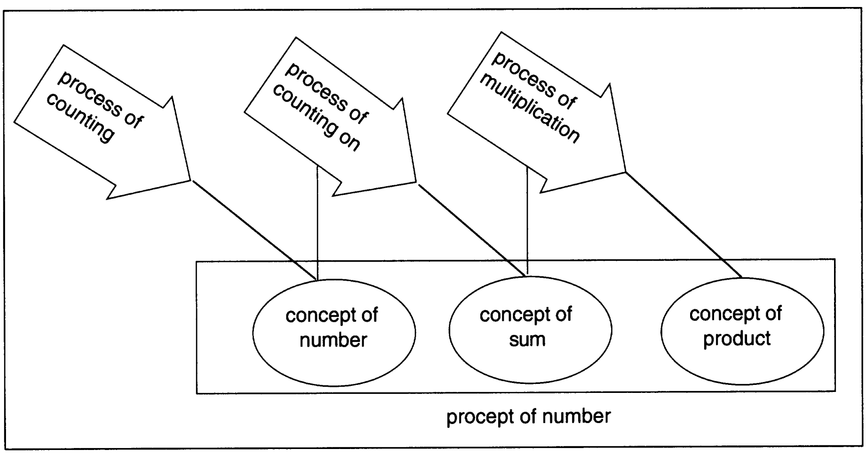


Figure 10. Collapse of hierarchy into operations on numbers.

We hypothesize that this is the development by which a more able thinker develops a flexible relational understanding in mathematics, which is seen as a meaningful relationship between notions at the same level, whereas the less able are faced with a hierarchical ladder that is more difficult to climb. We contend that children across a wide spectrum of performance face this challenge at each stage of encapsulation and that at each stage more children fail.

This provides an insight into why the practicing expert sees mathematics as such a simple subject and may find it difficult to appreciate the difficulties faced by the novice. As Thurston indicated in our earlier quotation, it is the compression of mathematical ideas that makes them so simple. As proceptual thinking grows in conceptual richness, procepts can be manipulated as simple symbols at a higher level or opened up to perform computations, to be decomposed and recomposed at will. Such forms of thinking become entirely unattainable for the procedural thinker who fails to develop a rich proceptual structure.

For unto everyone that hath shall be given and he shall have abundance: but from him that hath not shall be taken even that which he hath (Matthew, 25:29).

EXAMPLES FROM OTHER AREAS OF MATHEMATICS

Our empirical evidence in this paper has concentrated on simple arithmetic. However, other research can also be reinterpreted in proceptual terms. We have evidence that the lack of formation of the procept for an algebraic expression causes difficulties for pupils who see the symbolism representing only a general procedure for computation: an expression such as $2 + 3x$ may be conceived as a process that cannot be carried out because the value of x is not known (Tall & Thomas, 1991). We have evidence that the conception of a trigonometric ratio only as a process of calculation (opposite over hypotenuse) and not a flexible procept causes difficulties in trigonometry (Blackett, 1990; Blackett & Tall, 1991). In both of these cases we have evidence that the use of the computer to carry out the process, thus enabling the learner to concentrate on the product, significantly improves the learning experience. The difference between ratio and rate also has an obvious interpretation in terms of procept, where ratio is a process and rate the mathematical object produced by that process.

The case of the function concept, where $f(x)$ in traditional mathematics represents both the process of calculating a specific value of x and the concept of function for general x, is another example where the method of conceiving a function as an encapsulated object causes great difficulty (Sfard, 1989). There is evidence (Schwingendorf, Hawks, & Beineke, 1992) that the programming of the function as a procedure whose name may also be used as an object significantly improves understanding of function as a procept.

The limit concept is also a procept, but of a subtly different kind. The symbolism for limit represents both the process of tending to a limit, "as $n \to \infty$ so $s_n \to s$" or "$\lim_{n \to \infty} s_n = s$," and the value of the limit "$s = \lim_{n \to \infty} sn$." As Cornu (1981, 1983) showed, this causes a problem for students because there is no explicit procedure to calculate the limit; instead, it has to be computed by indirect means using general theorems on limits that may not be adequate to compute the precise value.

We therefore are confident that the notion of procept, with its ability to evoke process or product, offers an insightful analysis of success and failure in the process of learning mathematics. The subject has a spiraling complexity that more successful students compress by using symbols both as manipulable objects and as triggers to evoke mathematical processes. Meanwhile the less successful students eventually become trapped in procedural cul-de-sacs as the subject—for them—grows ever more complex.

REFERENCES

Baroody, A. J., & Ginsburg, H. P. (1986). The relationship between initial meaningful and mechanical knowledge of arithmetic. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 75–112). Hillsdale, NJ: Lawrence Erlbaum.

Blackett, N. (1990). Developing understanding of trigonometry in boys and girls using a computer to link numerical and visual representations. Unpublished doctoral dissertation, University of Warwick, U.K.

Blackett, N., & Tall, D. O. (1991). Gender and the versatile learning of trigonometry using computer software. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 144–151). Assisi, Italy.

Carpenter, T. P., (1986). Conceptual knowledge as a foundation for procedural knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 113–132). Hillsdale, NJ: Lawrence Erlbaum.

Carpenter, T. P., Hiebert, J., & Moser, J. M., (1981). Problem structure and first grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education, 12,* 27–39.

Carpenter, T. P., Hiebert, J., & Moser, J. M. (1982). Cognitive development and children's solutions to verbal arithmetic problems. *Journal for Research in Mathematics Education, 13,* 83–98.

Cornu, B. (1981). Apprentissage de la notion de limite: modèles spontanés et modèles propres [Learning the notion of limit: spontaneous models and formal models]. In C. Comiti & G. Vernaud (Eds.), *Proceedings of the Fifth International Conference for the Psychology of Mathematics Education,* (pp. 322–326). Grenoble, France.

Cornu, B. (1983). Apprentissage de la notion de limite: conceptions et obstacles [Learning the notion of limit: conceptions and obstacles]. Thèse de Doctorat, Grenoble, France.

Davis, R. B. (1983). Complex mathematical cognition. In H. P. Ginsburg (Ed.), *The development of mathematical thinking,* (pp. 254–290). New York: Academic Press,

Dienes, Z. P. (1960). *Building up mathematics.* London: Hutchinson Educational.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking,* (pp. 95–123). Dordrecht, The Netherlands: Kluwer.

Fuson, K. C., (1982). An analysis of the counting-on solution procedure in addition. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 67–81). Hillsdale, NJ: Lawrence Erlbaum.

Fuson, K., & Fuson, A. M. (1992). Instruction supporting children's counting-on for addition and counting-up for subtraction. *Journal for Research in Mathematics Education, 23,* 52–78.

Fuson, K. C., & Hall, J. W. (1983). The acquisition of early number word meanings: A conceptual analysis and review. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp.49–107). New York: Academic Press.

Fuson, K. C., Richards, J., & Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In C. Brainerd (Ed.), *Progress in cognitive development research: Vol 1. Children's logical and mathematical cognition,* (pp. 33–92). New York: Springer Verlag.

Gelman, R., & Gallistel, C. R. (1986). *The child's understanding of number* (2nd ed.). Cambridge, MA: Harvard University Press.

Gelman, R., & Meck, E. (1986). The notion of principle: The case of counting. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 29–58). Hillsdale, NJ: Lawrence Erlbaum.

Gray, E. M. (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics, 22,* 551–574.

Gray, E. M., & Tall, D. O. (1991). Duality, ambiguity and flexibility in successful mathematical thinking. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 72–79). Assisi, Italy.

Greeno, J. (1983). Conceptual entities. In D. Genter & A. L. Stevens (Eds.), *Mental models* (pp. 227–252). Hillsdale, NJ: Lawrence Erlbaum.

Herscovics, N., & Bergeron, J. C. (1983). Models of understanding. *Zentralblatt für Didaktik der Mathematik, 83,* 75–83.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case for mathematics* (pp. 1–28). Hillsdale, NJ: Lawrence Erlbaum.

Kaput, J. (1982, April). *Differential effects of the symbol systems of arithmetic and geometry on the interpretation of algebraic symbols.* Paper presented at the Annual Meeting of the American Educational Research Association, New York.

Klahr, D., & Wallace, J. G. (1976). *Cognitive development; an information processing view.* Hillsdale, NJ: Lawrence Erlbaum.

Linchevski, L., & Sfard, A. (1991). Rules without reasons as processes without objects: the case of equations and inequalities. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education* (pp. 317–324). Assisi, Italy.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: Author.

National Curriculum (1989). *The national curriculum—from policy to practice.* London: H.M.S.O.

Piaget, J. (1952). *The child's conception of number* (C. Gattegno & F. M. Hodgson, Trans.). London: Routledge & Kegan Paul.

Piaget, J. (1972). *The principles of genetic epistemology* (W. Mays, Trans.). London: Routledge & Kegan Paul.

Piaget, J. (1985). *The equilibration of cognitive structures.* Cambridge, MA: Harvard University Press.

Schwingendorf, K., Hawks, J., & Beineke, J. (1992). Horizontal and vertical growth of the student's concept of function. In E. Dubinsky & G. Harel, (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 133–149). MAA Notes, 25. Mathematical Association of America.

Secada, W. G., Fuson, K. C., & Hall, J. W. (1983). The transition from counting-all to counting-on in addition. *Journal for Research in Mathematics Education, 14,* 47–57.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. In G. Vergnaud, J. Rogalski, and M. Artigue (Eds.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education* (pp. 151–158). Paris, France.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1–36.

Skemp, R. R. (1979). *Intelligence, learning and action.* London: Wiley.

Sinclair, H., & Sinclair, A. (1986). Children's mastery of written numerals and the construction of basic number concepts. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 59–74). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P., von Glasersfeld, E., Richard, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory and application.* New York: Praeger Scientific.

Steinberg, R. M. (1985). Instruction in derived fact strategies in addition and subtraction. *Journal for Research in Mathematics Education, 16,* 337–355.

Tall D. O., & Thomas M. O. J. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics, 22,* 125–147.

Thurston, W. P. (1990). Mathematical education. *Notices of the American Mathematical Society, 37,* 844–850.

Wagner, S. H., & Walters, J. (1982). A longitudinal analysis of early number concepts. In G. E. Freeman (Ed.), *Action and thought: From sensorimotor schemes to symbolic operations* (pp. 137–161). New York: Academic Press.

Wohlwill, J., & Lowe, R. (1962). Experimental analysis of the development of the conservation of number. *Child Development, 33,* 133–167.

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LOOKING BACK

As of January 1971, there were 3162 individual and 495 institutional subscriptions.

Thomas Romberg, "Editorial Comment"
JRME, 1971, *3*, p. 179

Editor's Note. As of January 1994, there were 8550 individual and 1775 institutional subscriptions.